

ON THE FIBRATION OF AUGMENTED LINK COMPLEMENTS

DARLAN GIRÃO

ABSTRACT. We study the fibration of flat augmented link complements: simple combinatorial conditions on the diagram imply that these links fiber. We further show that certain surgeries on these links produce fibered manifolds. This is then used to prove that within a very large class of links, called locally alternating augmented links, every link is fibered.

1. INTRODUCTION

Let K be an oriented link in S^3 . By a Seifert surface S we mean an orientable spanning surface for K , i.e., $\partial S = K$ and the orientation of S agrees with that of K . It is known that every oriented link has a Seifert surface (see for instance [Ro]). We say the link K is *fibered* if $S^3 - K$ has the structure of a surface bundle over the circle, i.e., if there exists a Seifert surface S such that $S^3 - K \cong (S \times [0, 1])/\phi$, where ϕ is a homeomorphism of S . In this case we abuse terminology and say S is a *fiber* for K .

Deciding whether or not a link K is fibered, or even virtually fibered, is in general a very hard problem. In the early 60's Murasugi ([Mu]) proved that an alternating link is fibered if and only if its reduced Alexander polynomial is monic. Stallings [St] proved that a link K is fibered if and only if $\pi_1(S^3 - K)$ contains a finitely generated normal subgroup whose quotient is \mathbb{Z} . Murasugi's work is constructive, but restricts to alternating links. Stallings' result is very general, but usually very hard to verify. In [Ga] Gabai proved that if a Seifert surface S can be decomposed as the *Murasugi sum* of surfaces S_1, \dots, S_n , then S is a fiber if and only if each of the surfaces S_i is a fiber. Goodman-Tavares ([GT]) showed that under simple conditions imposed on certain spanning surfaces, it is possible to decide whether or not these surfaces are fibers for pretzel links. Leininger ([Le]) provided the first examples of virtually fibered but not fibered knots. Walsh ([Wa]) showed that all two-bridge knots and links are virtually fibered. She also showed that spherical Montesinos knots and links are virtually fibered. Very recently Futer-Kalfagianni-Purcell ([FKP1], theorem 5.11) introduced

a new method for deciding whether or not a given spanning surface is fiber for a link K . From a diagram of the link they construct an associated surface (called A -state surface) and a certain graph. They show that this surface is a fiber if and only if the corresponding graph is a tree.

On these notes we will be mainly concerned with the fibration of three classes of links: *augmented links*, *locally alternating augmented links*, and links obtained from augmented links by performing certain surgeries.

Augmented links have played a central role in several recent developments in 3-manifold topology. Lackenby and Agol-Thurston ([La1]) used them to estimate volumes of alternating link complements. Futer-Kalfagianni-Purcell ([FKP]) used them to obtain diagrammatic volume estimates of many knots and links. Futer-Purcell ([FP]) also used them to prove that if K is a link with a twist-reduced diagram with at least 4 twist regions and at least 6 crossings per twist region, then every non-trivial Dehn filling K is hyperbolic. Their combinatorial argument further implies that every link with at least 2 twist regions and at least 6 crossings per twist region is hyperbolic and give a lower bound for the genus of K . Cheesebro-DeBlois-Wilton ([CDW]) proved that hyperbolic augmented links satisfy the virtual fibering conjecture. However they did not provide any information on the virtual fibers.

Here we provide combinatorial conditions on the diagram, which imply that very often augmented links fiber and explicitly exhibit their fibers. We also show that when this is the case, then certain Dehn surgeries on these links produce fibered manifolds. This last result is then used to show that within the class of locally alternating augmented links every link is fibered, and explicitly exhibit their fibers.

We next define these classes we'll be working with and state the main results.

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2. AUGMENTED LINKS, LOCALLY ALTERNATING AUGMENTED LINKS AND MAIN RESULTS

The notion of *augmented links* was first introduced by Adams ([Ad1]) and further explored by Futer-Kalfagianni-Purcell ([FKP]), Futer-Purcell

([FP]) and Purcell ([Pu1, Pu2]). We recall it here. For more details see the very nice survey paper on augmented links by Purcell ([Pu]).

Let K be a link in S^3 with diagram $D(K)$. Regard $D(K)$ as a 4-valent graph in the plane. A *bigon region* is a complementary region of the graph having two vertices in its boundary. A string of bigon regions of the complement of this graph arranged end to end is called a *twist region*. A vertex adjacent to no bigons will also be a twist region. Encircle each twist region with a single unknotted component, called a *crossing circle*, obtaining a link J . $S^3 - J$ is homeomorphic to the complement of the link L obtained from J by removing all full twists from each twist region. The link L is called the augmented link associated to $D(K)$. The original link complement can be obtained from the link J by performing $1/n$ -Dehn filling on the crossing circles, for appropriate choices of n .

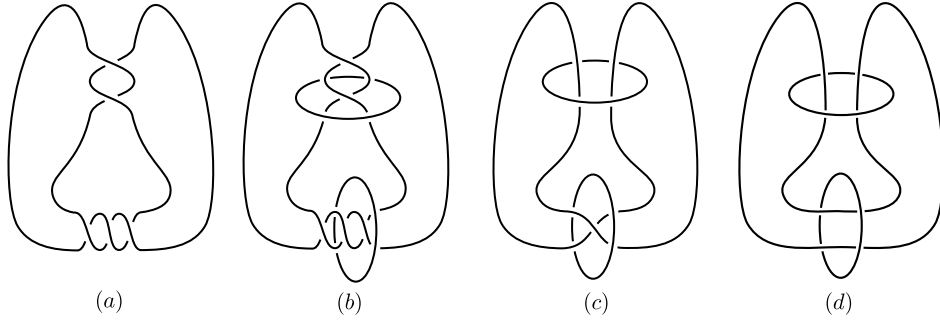


FIGURE 1. (a) Initial link K ; (b) link J obtained by adding crossing circles; (c) augmented link L ; (d) corresponding flat augmented link.

When all the twist regions in the diagram $D(K)$ have an even number of crossings, then all non-crossing circle components of the augmented link L will be embedded in the projection plane. We denote these types of links by *flat augmented links*.

Given the diagram of a flat augmented link L we construct a Seifert surface S_L , called *standard Seifert surface*, and a graph $G_B(L)$ (this will be done in section 4). We now state our main results.

Theorem A. *Let L be a flat augmented link. Then the standard Seifert surface S_L is a fiber if and only if the graph $G_B(L)$ is a tree.*

Performing ± 1 Dehn surgery on crossing circle components of links as above yield new ones which are again fibered.

Theorem B. *Let L be an flat augmented link such that the graph $G_B(L)$ is a tree. Then performing ± 1 Dehn surgery on crossing circle components yields a fibered link K .*

Given a flat augmented link L , one can construct a corresponding *locally alternating augmented link* L_a as follows: in each crossing circle change two of the crossings so that the crossings in the crossing circles are alternating. This is described in Figure 2. Note that the resulting link need not to be alternating.

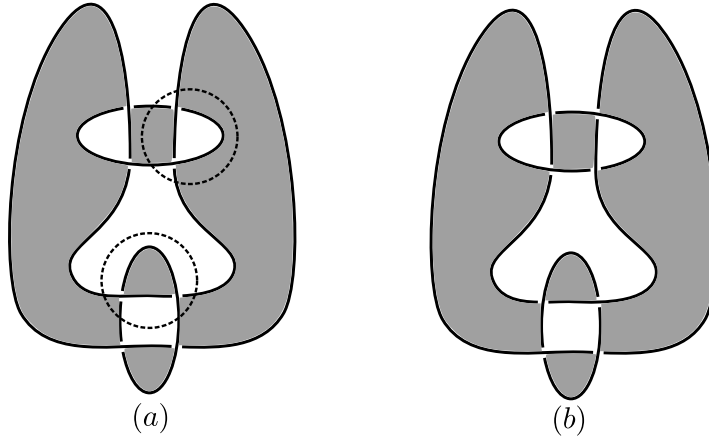


FIGURE 2. (a) Pairs of crossings yielding locally alternating link; (b) Canonical Seifert surface for resulting link.

We later show that every locally alternating augmented link L_a can be obtained from ± 1 surgery on the crossing circles of a flat augmented link \tilde{L} such that $G_B(\tilde{L})$ is a tree. This implies

Theorem C. *Let L_a be an locally alternating augmented link obtained from an flat augmented link L . Then L_a fibers.*

Remark 1. We remark that our methods and the surfaces and graphs we construct are very different from those in [FKP1]. However, it is very interesting that we are obtaining the same type of results: a manifold fibers given that a certain associated graph is a tree. We also note that very often fibration of the links considered here cannot be detected from their construction but is detected by ours (examples for the converse can also be exhibited).

The remainder of the paper is organized as follows: in section 3 we recall the notion of *Murasugi sum*. This will be used in the construction of the *standard surface*. In section 4 we set up the terminology

of standard surface and of the $G_B(L)$ graph needed for the remainder of the paper. In section 5 we prove theorem **A**. In section 6 we prove theorem **B**. Finally, in section 7 we use theorem **B** to prove theorem **C**.

3. MURASUGI SUM

In this section we recall the notion of Murasugi sum ([Ga1], [Le], [Mu]).

Definition 1. We say that the oriented surface T in S^3 is with boundary L is the Murasugi sum of the two oriented surfaces T_1 and T_2 with boundaries L_1 and L_2 if there exists a 2-sphere S in S^3 such bounding the balls B_1 and B_2 with $T_i \subset B_i$ for $i = 1, 2$, such that $T = T_1 \cup T_2$ and $T_1 \cap T_2 = D$ where D is a $2n$ -sided disk contained in S (see Figure 3).

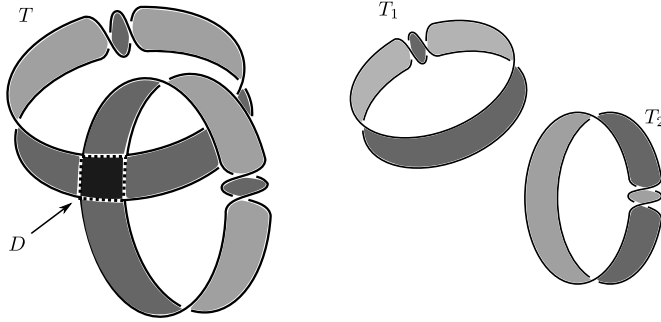


FIGURE 3.

The result concerning Murasugi sum we need is the following, due to Gabai ([Ga]).

Theorem 3.1. *Let $T \subset S^3$, with $\partial T = L$, be a Murasugi sum of oriented surfaces $T_i \subset S^3$, with $\partial T_i = L_i$, for $i = 1, 2$. Then $S^3 - L$ is fibered with fiber T if and only if $S^3 - L_i$ is fibered with fiber T_i for $i = 1, 2$.*

Remark 2. We abuse notation by saying that L is the Murasugi sum of L_1 and L_2 .

4. SET UP

The augmented links we consider are flat, i.e., there are no twists adjacent to crossing circles. We may isotope the link slightly so that in its diagram all crossing circles encircle locally vertical strings and so that

the overcrossings of the crossing circles are below the undercrossings (see Figure 4 (b)).

Our goal is to find a condition under which such links fiber. Recall that Stallings ([St]) provided a method for checking whether or not a given Seifert surface is a fiber for the complement of an oriented link.

Theorem 4.1 (Stallings). *Let $T \subset S^3$ be a compact, connected, oriented surface with nonempty boundary ∂T . Let $T \times [-1, 1]$ be a regular neighborhood of T and let $T^+ = T \times \{1\} \subset S^3 - T$. Let $f = \varphi|_T$, where $\varphi : T \times [-1, 1] \rightarrow T^+$ is the projection map. Then T is a fiber for the link ∂T if and only if the induced map $f_* : \pi_1(T) \rightarrow \pi_1(S^3 - T)$ is an isomorphism.*

Let L be a flat augmented link with the description as above. Orient planar components in the clockwise direction and crossing circle components in the counterclockwise direction, as seen from above the projecting plane (see Figure 4). Consider the Seifert surface given by the Seifert algorithm. This Seifert surface will be called the *canonical Seifert surface* and the above orientation for L the *canonical orientation*.

Consider the diagram of L as a planar graph. This graph divides the plane into regions which are checkerboard colored. The unbounded region is colored white and the other regions are colored accordingly. There are three types of white regions:

Type A regions, bounded by those crossing circles that bound two white regions;

Type B regions, bounded by those crossing circles that bound a single white region;

Type C regions, not bounded by crossing circles.

Note that type A regions come in pairs. We will denote the above regions by $A_{11}, A_{12}, A_{21}, A_{22}, \dots, A_{p1}, A_{p2}, B_1, \dots, B_q, C_1, \dots, C_r$ respectively. We denote the unbounded white region by C_0 .

A crossing circle bounding a type B region B_i will be called a *B-circle*, also denoted by B_i . Note that given the diagram for L , as described above, there will be a type C region to the right and another one to the left of every B-circle. We say the B-circle is *adjacent* to these regions. A crossing circle bounding a pair of type A regions A_{i1}, A_{i2} will be called a *A-circle* and denoted by A_i . Note that there will be one type C region below and one above of every A-circle. We say the A-circle is *adjacent* to them. Let $G_B(L)$ be the graph obtained from the diagram of L as follows

Vertices are type C regions;

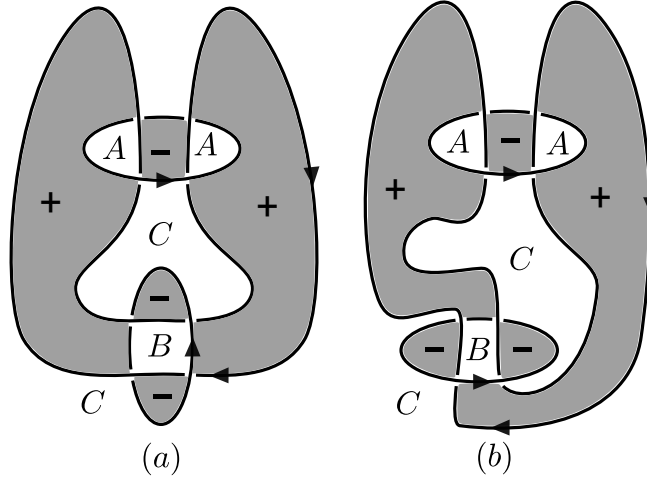


FIGURE 4. Canonical Seifert surface for augmented link and corresponding white regions determined by the diagram

An **edge** joins C_i and C_j if there is a B -circle adjacent to both of them simultaneously.

From the canonical Seifert surface with canonical orientation we construct the *standard Seifert surface*, denoted S_L , as follows:

1. Remove the portion of the link and of the canonical surface contained in the interior of a sphere around A -circles.
2. For each A -circle removed, add an untwisted band as described in Figure 5.
3. Perform Murasugi sum of two Hopf bands along the untwisted band added on 2 (see Figure 6).

Figure 7 illustrates two examples of the construction of the graph $G_B(L)$ from the diagram of the link L .

5. PROOF OF THEOREM A

Theorem A. *Let L be a flat augmented link. Then the standard Seifert surface S_L is a fiber if and only if the graph $G_B(L)$ is a tree.*

Note that the the standard surface S_L is obtained as the Murasugi sum of a collection of Hopf bands together with a leftover surface. Each of these Hopf bands is a fiber for the complement of their boundary Hopf links. Denote by $S_{L'}$ the resulting leftover surface and by L' its boundary link. L' is itself a flat augmented link in which all crossing circles are B -circles. Observe also that, by construction, $G_B(L) =$

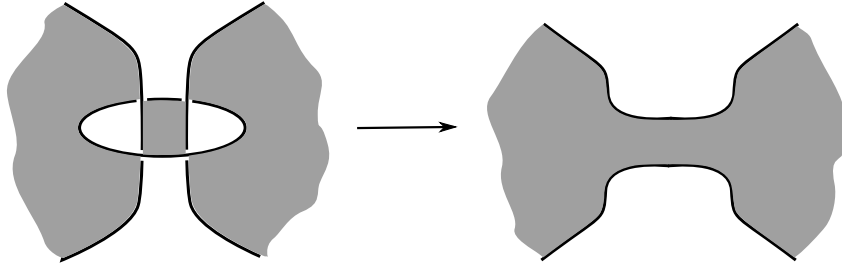


FIGURE 5. Replacing A -circle by untwisted band (steps 1 and 2 above)

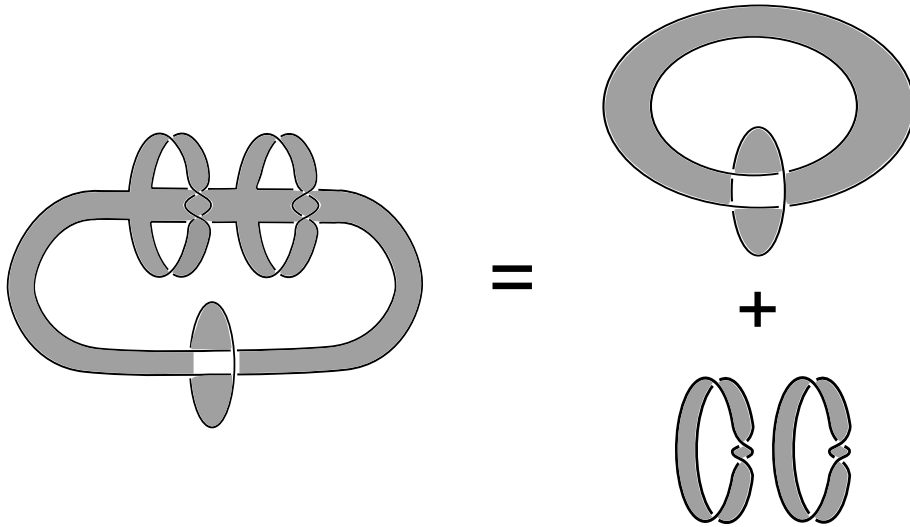


FIGURE 6. Obtaining A -circle from Murasugi sum of two bands.

$G_B(L')$. In view of Theorem 3.1 we need to find conditions under which the surface $S_{L'}$ is a fiber for the link L' . This is the content of

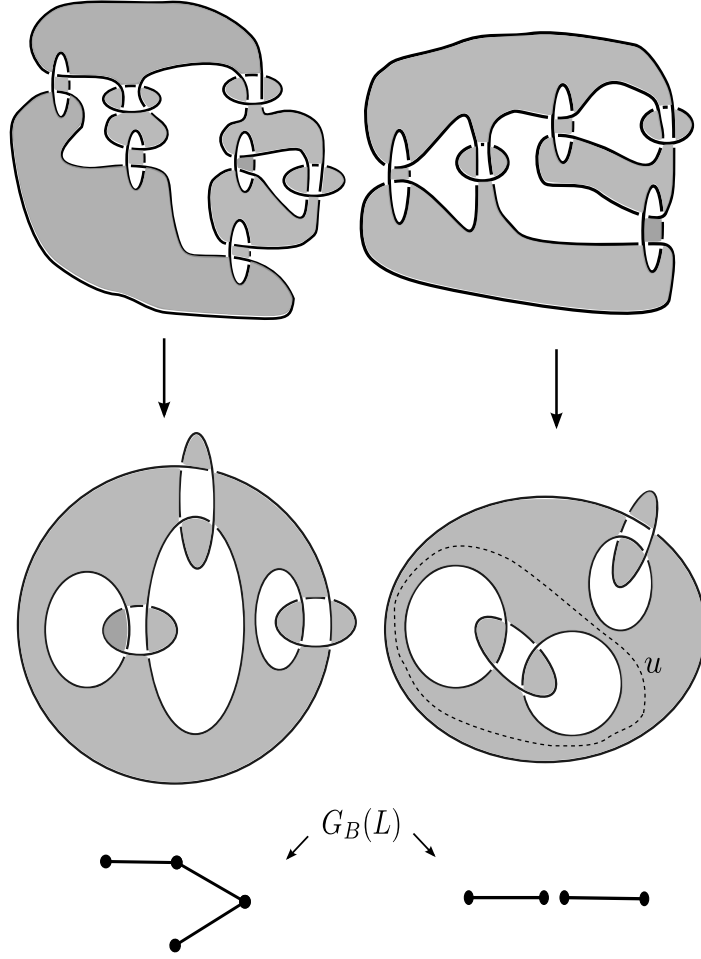
Lemma 5.1. *The oriented link L' fibers with fiber $S_{L'}$ if and only if its associated graph $G_B(L')$ is a tree.*

This lemma concludes the proof of Theorem A.

Remark 3. Note that, for L' , its canonical and standard Seifer surface coincide.

We now proceed to prove the lemma.

Proof of lemma 5.1. Observe first that the link L' is itself a flat augmented link, obtained from L by removing its A -circles and adding

FIGURE 7. Original link; replace A -circles; associated graph.

untwisted bands. The diagram of L' divides the plane into type B and type C regions.

Observe also that the fundamental group of the surface $S_{L'}$ is free with a generating set given by the loops on the surface around white regions. This can be easily seen since a regular neighborhood of this surface is a handlebody. Let B_1, \dots, B_q and C_1, \dots, C_r be the type B and C regions given by the diagram of L' . Denote the loops on $S_{L'}$ around these regions by $u_{b_1}, \dots, u_{b_q}, u_{c_1}, \dots, u_{c_r}$ respectively. They will be oriented counterclockwise.

The fundamental group of the complement $S^3 - S_{L'}$ of the above Seifert surface is also free with generating set given by the loops going through white regions perpendicular to the projecting plane in $S^3 - S_{L'}$ (this is true since a regular neighborhood of this surface is a handlebody).

These loops are denoted by x_{b_1}, \dots, x_{b_q} and x_{c_1}, \dots, x_{c_r} , according to the type of region. All these loops are to be oriented from above to below the projecting plane.

The surface $S_{L'}$ is two sided and our convention throughout is that from above the diagram of L' we see the “−” side of the pieces bounded by a crossing circle (see Figure 4). Let $S_{L'}^+$ be the copy of $S_{L'}$ in $S^3 - S_{L'}$ parallel to $S_{L'}$ obtained from $S_{L'}$ by pushing it in the “+” direction. This is formally defined by the map $f : S_{L'} \rightarrow S^3 - S_{L'}$ described in Theorem 4.1.

With this in mind it is not hard to describe the induced map f_* (see Figure 8). We have:

$$\begin{cases} u_{b_i} \mapsto x_{c_m} x_{c_n}^{-1} \\ u_{c_m} \mapsto x_{b_i} x_{c_m}^{-1} w_m, \text{ where } w_m \text{ is a word without the letter } x_{b_i}; \\ u_{c_n} \mapsto x_{c_n} x_{b_i}^{-1} w_n, \text{ where } w_n \text{ is a word without the letter } x_{b_i}. \end{cases}$$

Remark 4. If a B -circle is adjacent to the unbounded region C_0 and another region C_m , then we have $u_{b_i} \mapsto x_{c_m}^{\pm 1}$.

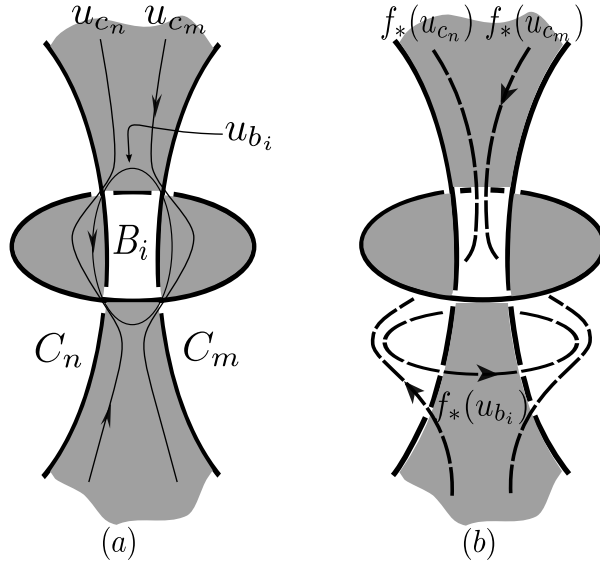


FIGURE 8. (a) Loops u_{b_i}, u_{c_m} and u_{c_n} ; (b) Their images under f_* .

The strategy now is as follows: by Theorem 4.1 $S_{L'}$ is a fiber iff the map $f_* : \pi_1(S_{L'}) \rightarrow \pi_1(S^3 - S_{L'})$ is an isomorphism. We show that if $G_B(L')$ is not a tree then f_* is not injective. When $G_B(L')$ is a tree we show that f_* is surjective and therefore an isomorphism (recall that free groups are Hopfian).

First assume $G_B(L')$ is not a tree

Case 1. $G_B(L')$ has an isolated vertex: For this case, we have an isolated type C region C_m , i.e., C_m is not adjacent to any B -circles. It is easy to see that the loop u_{c_m} is not trivial in $S_{L'}$ but $f_*(u_{c_m})$ is trivial in $S^3 - S_{L'}$.

Case 2. $G_B(L')$ has two or more connected components:

Let C_1, \dots, C_{n+1} represent the vertices of the component not containing the vertex C_0 . Let $u \subset S_{L'}$ be the loop around the regions C_1, \dots, C_{n+1} (see Figure 7). This loop is obviously non-trivial in $S_{L'}$. However we see that $f_*(u)$ is trivial in $S^3 - S_{L'}$.

Case 3. $G_B(L')$ contains a nontrivial loop: Let $C_1, \dots, C_n, C_{n+1} = C_1$ represent the vertices and B_1, \dots, B_n the edges of this loop. We have

$$u_{b_i} \mapsto x_{c_i}^{-1} x_{c_{i+1}}$$

and thus

$$f_*(u_{b_1} u_{b_2} \cdots u_{b_n}) = (x_{c_1}^{-1} x_{c_2})(x_{c_2}^{-1} x_{c_3}) \cdots (x_{c_n}^{-1} x_{c_1}) = 1$$

The same argument holds if the loop contains the vertex corresponding to C_0 .

Next we show that if $G_B(L')$ is a connected tree then the surface $S_{L'}$ is a fiber. This relies again on Stallings fibering result.

Step 1. Let C_0 be the initial vertex and let C_1, \dots, C_k be the vertices connected to C_0 by edges represented by the B -circles B_1, \dots, B_n . We have

$$u_{b_i} \mapsto x_{c_i}^{\pm}$$

This allows us to define an inverse for x_{c_i} in terms of u_{b_i} .

Step 2. Let C_{11}, \dots, C_{1k_1} be connected to C_1 by edges B_{11}, \dots, B_{1k_1} . Note that, since $G_B(L')$ is a tree, $C_{1i} \neq C_j$ for $i = 1, \dots, k_1$ and $j = 0, \dots, k$. We have

$$u_{b_{1j}} \mapsto x_{c_1}^{\pm 1} x_{c_{1j}}^{\mp 1}$$

This defines an inverse for $x_{c_{1i}}$ in terms of $u_{b_{1i}}$ and u_{b_1} .

Step 3. Repeat Step 2 for C_2, \dots, C_k . These C_i 's are the ones defined in Step 1. This defines an inverse for $x_{c_{ji}}$ in terms of $u_{b_{ji}}$ and u_{b_j} .

Step 4. Inductively define inverses for all the x'_c s in terms of the u'_b s. Note that, for this step to work, it is fundamental that $G_B(L')$ is a tree.

Now we need to find inverses for the x_b 's.

Step 5. From

$$u_{c_m} \mapsto x_{c_m}^{-1} x_{b_i} w_m, w_m \text{ word without the } x_{b_i} \text{ letter}$$

$$u_{c_n} \mapsto x_{c_n} x_{b_i}^{-1} w_n, w_n \text{ word without the } x_{b_i} \text{ letter}$$

we will show how to write the x_b 's in terms of u_b 's and u_c 's.

Since $G_B(L')$ is a tree, each C -region corresponding to a terminal vertex is adjacent to a single B -circle. Suppose C_m is such a region. From $u_{c_m} \mapsto x_{c_m}^{-1} x_{b_i}$ we can write x_{b_i} in terms of u_b 's and u_c 's. Repeat this process to all the terminal vertices. Inductively we can write the x_b 's in terms of u_b 's and u_c 's.

□

6. DEHN SURGERIES ON CROSSING CIRCLES

In this section we prove

Theorem B. *Let L be an flat augmented link such that the graph $G_B(L)$ is a tree. Then performing ± 1 Dehn surgery on crossing circle components yields a fibered link K .*

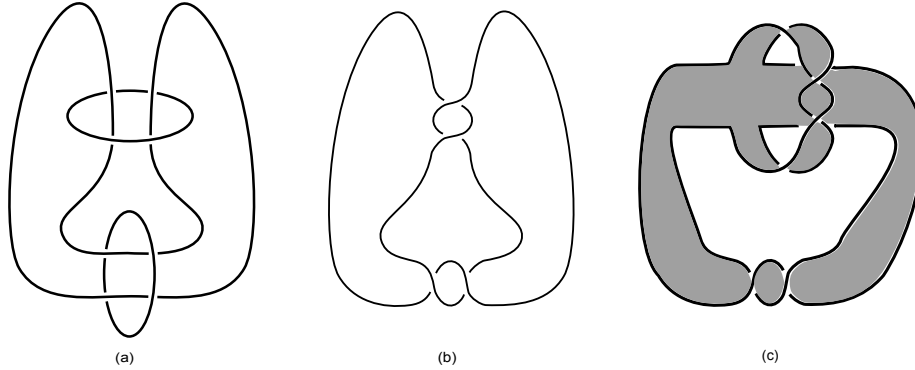


FIGURE 9. (a) Original link; (b) ± 1 surgery on crossing circles; (c) Standard Seifert surface for resulting link.

Proof. Performing ± 1 surgery on a A -circles yields a fibered link. To see this consider the link L' constructed as in the last section. Recall that when L fibers with fiber S_L , then L' fibers with fiber $S_{L'}$. A -circles in L correspond to the Murasugi sum of two Hopf bands with L' . It should be easy to see that ± 1 surgery on a A -circle corresponds to the sum of a single Hopf band. Therefore, by theorem 3.1, the statement is true for surgeries on A -circles.

The proof of the statement for B -circles is similar to the proof that $S_{L'}$ is a fiber for L' . Let K' be the link obtained from L' by performing

± 1 Dehn surgery on the B -circles. We consider the standard Seifert surface $S_{K'}$ obtained from the Seifert algorithm applied to the link K' (see Figure 10). We now describe the map $f_* : \pi_1(S_{K'}) \rightarrow \pi_1(S^3 - S_{K'})$. First we introduce some conventions: by considering the diagram of K' as a 4-valent graph, we may denote the loops around white regions of $S_{K'}$ by $\{u_c\}$, oriented as before. They are generators for $\pi_1(S_{K'})$. Similarly we have corresponding generators $\{x_c\}$ for $\pi_1(S^3 - S_{K'})$. We see that we may associate to $S_{K'}$ a tree $G_B(K')$ identical to $G_B(L')$ (see Figure 10). The *node* of the tree is the vertex corresponding to the unbounded C region C_0 . We say this vertex is at *level 0*. Let C_{11}, \dots, C_{1s} be the vertices adjacent to C_0 . We say these edges are at *level 1*. Recursevely define vertices at higher levels. f_* is described as follows:

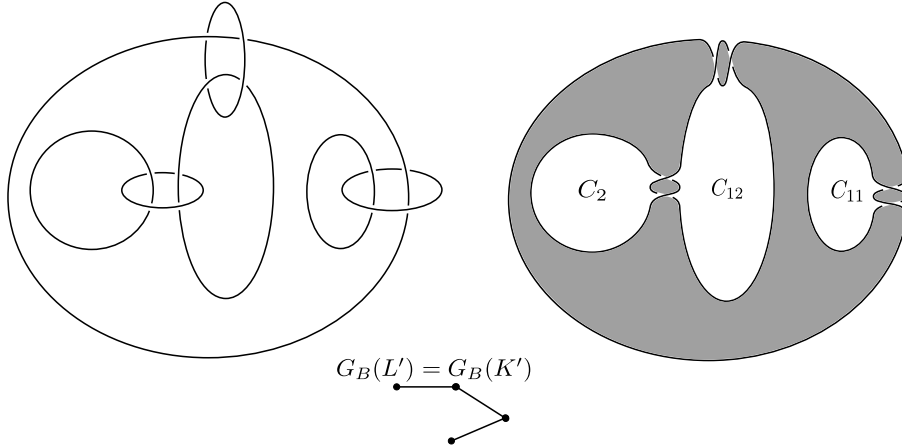


FIGURE 10. Left: link L' ; Right: K' and its standard Seifert surface.

1. For a terminal vertex, say C_m , on $G_B(K')$ we have: $u_{c_m} \mapsto w = x_{c_m} x_{c_{m-1}}^{-1}$ or $u_{c_m} \mapsto w' = x_{c_{m-1}} x_{c_m}^{-1}$, where C_{m-1} is the vertex one level lower than C_m and adjacent to it.
2. Let C_1 be a vertex at level 1 (adjacent to C_0). Suppose C_1 is also a terminal vertex. We have $u_{c_1} \mapsto x_{c_1}^{\pm 1}$.
3. If C_1 is not a terminal vertex, let C_{21}, \dots, C_{2s} be the vertices at level 2 adjacent to it. We have $u_{c_1} \mapsto x_{c_1}^{\pm 1} w_1 w_2 \dots w_s$, where the w_i are of one the forms $w_i = x_{c_{2i}} x_{c_1}^{-1}$ or $w_i = x_{c_1} x_{c_{2i}}^{-1}$.
3. An intermediate vertex (not a terminal vertex nor adjacent to C_0) C_{k-1} , say at level $k-1$, is adjacent to vertices C_{k1}, \dots, C_{ks} at level k and to a single vertex C_{k-2} at level $k-2$. We have

$u_{c_{k-1}} \mapsto (x_{c_{k-1}} x_{c_{k-2}}^{-1})^{\pm 1} w_1 w_2 \cdots w_s$, where the w_i are of one the forms $w_i = x_{c_{k-1}} x_{c_{ki}}^{-1}$ or $w_i = x_{c_{ki}} x_{c_{k-1}}^{-1}$.

Denote by $\{u_c\}_c$ the set of generators for the free group $\pi_1(S_{K'})$ and by $\{x_c\}_c$ the set of generators for $\pi_1(S^3 - S_{K'})$. Note that f_* maps the set $\{u_c\}_c$ to $\{y_c\}_c$, where y_c is a word on the letters x_c . Given the tree structure on $G_B(K')$, it is not hard to see from the above description of f_* how to obtain $\{y_c\}_c$ from $\{x_c\}_c$ by a sequence of Nielsen transformations (see example below). Therefore the map $f_* : \pi_1(S_{K'}) \longrightarrow \pi_1(S^3 - S_{K'})$ is an isomorphism. \square

Example 1. For the example in Figure 10 we have:

$$u_{c_{11}} \mapsto x_{c_{11}}, u_{c_{12}} \mapsto x_{c_{12}} x_{c_2} x_{c_{12}}^{-1}, u_{c_2} \mapsto x_{c_{12}} x_{c_2}^{-1}$$

and the sequence of Nielsen transformations is given by

$$\begin{cases} x_{c_{11}} \mapsto x_{c_{11}} & \mapsto x_{c_{11}} \\ x_{c_{12}} \mapsto x_{c_{12}} & \mapsto x_{c_{12}} (x_{c_{12}} x_{c_2}^{-1})^{-1} \\ x_{c_2} \mapsto x_{c_{12}} x_{c_2}^{-1} & \mapsto x_{c_{12}} x_{c_2}^{-1} \end{cases}$$

7. LOCALLY ALTERNATING AUGMENTED LINKS

Given a flat augmented link L , we construct a corresponding *locally alternating augmented link* L_a as described in section 2 (see Figure 2). Regarding the fibration of the oriented link L_a , we get a stronger statement.

Theorem C. *Let L_a be a locally alternating augmented link obtained from an flat augmented link L . Then L_a fibers.*

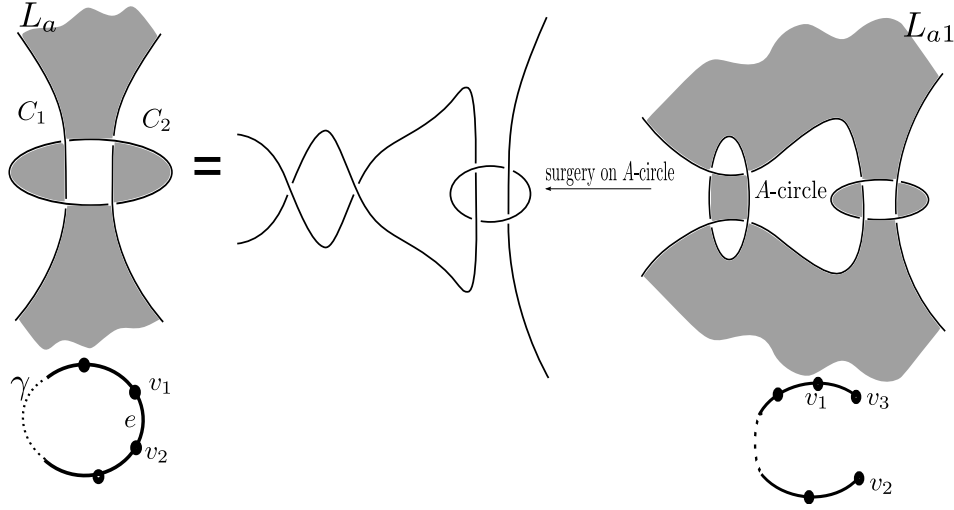
This theorem follows immediately from

Lemma 7.1. *Let L_a be a locally alternating augmented link obtained from a flat augmented link L . Then L_a is obtained from ± 1 surgeries on the crossing circles of a flat augmented link \tilde{L} such that $G_B(\tilde{L})$ is a tree.*

Proof of lemma. Just as for L , construct the standard Seifert surface for L_a , as well as the corresponding graph $G_B(L_a)$. Given L_a , suppose $G_B(L_a)$ is not a tree.

Step 1. Eliminate nontrivial loops in $G_B(L_a)$:

Choose a loop γ in $G_B(L_a)$ and an edge e on this loop (a B -circle on L_a) connecting vertices v_1, v_2 corresponding to regions C_1, C_2 . Figure 11 shows how to obtain L_a from ± 1 sugery on a A -circle of a new link L_{a1} . The relationship between $G_B(L_a)$ and $G_B(L_{a1})$ is that $G_B(L_{a1})$ is

FIGURE 11. Removing the loop γ from $G_B(L_a)$.

obtained from $G_B(L_a)$ by adding a new vertex v_3 in the interior of e and breaking the loop γ at v_3 .

Step 2. Join disconnected components of $G_B(L_a)$:

Consider two vertices v_1, v_2 on $G_B(L_a)$ not in the same component but such that their corresponding C -regions C_1, C_2 are adjacent to a A -circle. Figure 12 shows how to obtain L_a from ± 1 surgery on a B -circle of a new link L_{a2} . The relationship here is that $G_B(L_{a2})$ is obtained from $G_B(L_a)$ by joining v_1, v_2 by a new edge e .

Step 3. Repeat the steps above, as needed, until we obtain a link \tilde{L}_0 such that $G_B(\tilde{L}_0)$ is a tree. Note that some of its crossing circles are alternating and some are not. L_a is obtained from \tilde{L}_0 by a sequence of ± 1 surgeries on its flat crossing circles (these ones are the ones introduced in the steps above).

The conclusion now follows from the observation that alternating crossing circles can be obtained from ± 1 surgeries on a pair of flat crossing circles, as described in Figure 13. The desired link \tilde{L} is the link obtained from \tilde{L}_0 by replacing alternating crossing circles by a pair of non-alternating ones. Note that since $G_B(\tilde{L}_0)$ is a tree, $G_B(\tilde{L})$ is also a tree.

□

8. RELATED QUESTIONS

Several questions arise from the the results above.

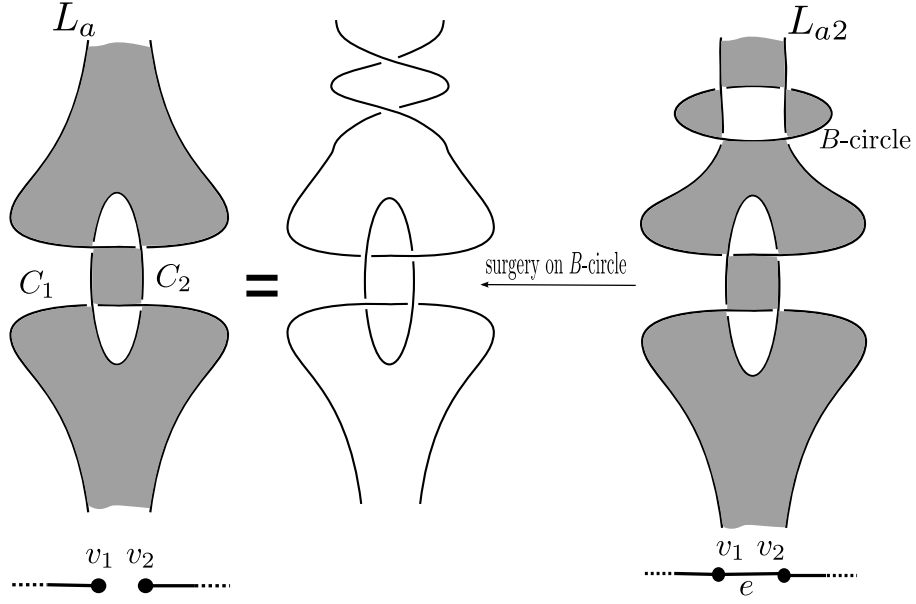
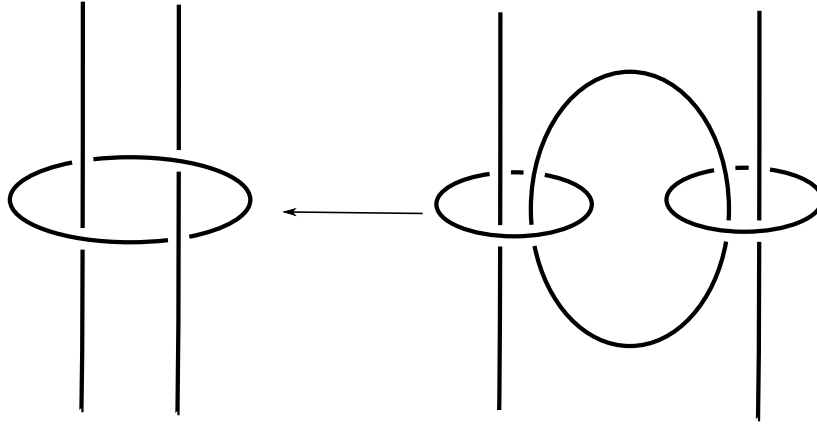
FIGURE 12. Joining vertices v_1 and v_2 .

FIGURE 13. Obtaining alternating crossing circles from surgeries on non-alternating ones.

Question 1. Is there a result similar to that of Theorem 2 for general augmented links, i.e., those with twists adjacent to crossing circles?

A big difference between the general case to the flat case is that in the presence of twists, Seifert surfaces become more complicated and Stallings's theorem is not readily verifiable. One could at least expect virtual fibration.

Question 2. Let \bar{L} be a general augmented link and L its associated flat augmented link. Assume $G_B(L)$ is a tree. Is it true that $S^3 - \bar{L}$ virtually fibers?

One possible approach is to show $S^3 - \bar{L}$ and $S^3 - L$ are commensurable.

Recall that Chesebro-DeBlois-Wilton proved that hyperbolic augmented link complements virtually fiber. One may be interested in finding more about the fibered covers of such link complements. We provide a rich class of augmented link which fiber and found fibers explicitly.

Problem 1. Obtain explicit information about fibered covers of such links.

Let K_1, \dots, K_r denote the crossing circle components of the flat augmented link L with $G_B(L)$ a tree. Let K be the link obtained from L by performing $1/n_i$ Dehn surgery on K_i , $i = 1, \dots, r$. In general one cannot expect the link K to be fibered. There are examples of knots arising in this fashion which are virtually fibered but not fibered. Twist knots provide a big class of such examples (see [Le]). Here we have shown that for surgery slopes ± 1 fibration persists. One may thus consider the following

Problem 2. Find combinatorial conditions on the diagram of K that imply K is a fibered link.

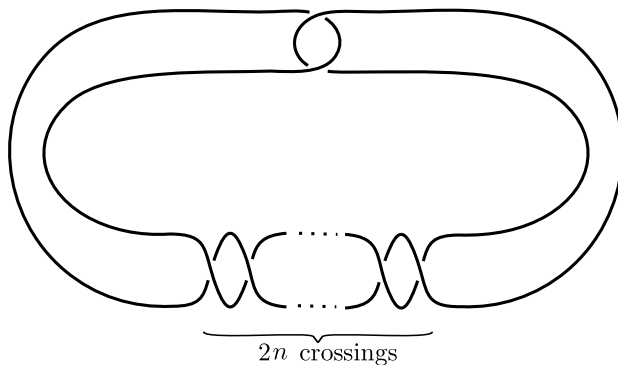


FIGURE 14. A twist knot (in general not fibered)

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DEPARTMENT OF MATHEMATICS,
THE UNIVERSITY OF TEXAS AT AUSTIN
E-mail: dgirao@math.utexas.edu